6. Homework Assignment

Dynamical Systems II

Bernold Fiedler, Hannes Stuke

http://dynamics.mi.fu-berlin.de/lectures/

due date: Thursday, November 27, 2014

Problem 1: Consider the upside-down pendulum

$$\ddot{\varphi} - \sin\varphi \;=\; 0.$$

Let

$$W^u_{loc} \;=\; \{(\varphi, \dot{\varphi}) = (\varphi, h(\varphi))\;;\; -\varepsilon < \varphi < \varepsilon\}$$

be the local unstable manifold at the equilibrium $\varphi = \dot{\varphi} = 0$. Determine the expansion

$$h(\varphi) \;=\; \sum_{k=0}^{N} h_k \varphi^k + \mathcal{O}(\varphi^{N+1})$$

up to order $N = 3$.

Hint: Use the invariance of W^s.

Extra credit: Determine the corresponding expansion for the damped pendulum

$$\ddot{\varphi} + \alpha\dot{\varphi} - \sin\varphi \;=\; 0$$

with $\alpha > 0$.

Problem 2: Consider the damped pendulum

$$\ddot{\varphi} + \alpha\dot{\varphi} - \sin\varphi \;=\; 0$$

with small $\alpha > 0$ and $\varphi \in \mathbb{R}$.

(i) Preferably by hand, sketch the global stable manifold of the equilibrium $\varphi = \dot{\varphi} = 0$.

(ii) How does the large-time behavior of trajectories differ above and below the stable manifold? Interpretation?

Problem 3: Consider a family $\Phi_n : X \to X$ of contractions on a Banach space X. Assume p-periodicity, i.e. $\Phi_{n+p} = \Phi_n$ for all $n \in \mathbb{N}_0$.

(i) Prove that there exists a unique $x_0 \in X$, such that the sequence

$$x_{n+1} := \Phi_n(x_n), \qquad n \geq 0,$$

is periodic, i.e. $x_{n+p} = x_n$ for all $n \in \mathbb{N}_0$.

(ii) Now assume $\Phi_n = \Phi$ does not really depend on $n \in \mathbb{N}_0$. Then any x_n from (i) is a p-periodic point under iterations of Φ: not only for $n = 0$, but also for any $0 \leq n < p$. How is that possible, in view of the uniqueness result of (i)?

Problem 4: Consider a diffeomorphism $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ with $\Phi(0) = 0$. Let

$$\begin{aligned} W^s &= \{x \in \mathbb{R}^N \mid \lim_{n\to\infty} \Phi^n(x) = 0\} \\ W^u &= \{x \in \mathbb{R}^N \mid \lim_{n\to\infty} \Phi^{-n}(x) = 0\} \end{aligned}$$

denote the stable and the unstable set of the origin. Find — if possible — an example and a counterexample for each of the following cases:

(i) W^s is an embedded submanifold.

(ii) W^s is closed.

(iii) $W^s \cap W^u$ consists of exactly two distinct points.

Extra credit: Find an example such that W^s is not even a manifold.

Reminder: W is a manifold of dimension M if it is locally homeomorphic to $\mathbb{R}^M$. W is an *embedded submanifold* of dimension M in $\mathbb{R}^N$ if for all $x \in W$ there exists a ball $x \in B_\varepsilon(x) \subset \mathbb{R}^N$ and a homeomorphism $h : B_\varepsilon(x) \to h(B_\varepsilon(x)) \subset B_1(0)$ such that $h(W \cap B_\varepsilon(x)) = (\{0\} \times \mathbb{R}^M) \cap h(B_\varepsilon(x))$. Here $B_1(0)$ is the unit ball in $\mathbb{R}^N$ centered at the origin.